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STABILITY OF PRETWISTED BEAMS IN CROSS BRACINGS

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New systems of pretwisted bars are suggested, and their stability under a conservative load is investigated. In this paper, closed-form relationships are obtained for the direct evaluation of critical loads of pretwisted cross bracings with different end connections and an arbitrary ratio of the dimensionless parameters of tension and compression elements. The equation for the critical loads is derived for tension and compression braces with different section properties, lengths, and axial loading. It is found that the critical load of a twisted brace is higher than that of a nontwisted brace. Parametric solutions are graphically displayed to clarify the distinct behavior, including the boundary separating symmetric and antisymmetric modes of buckling.

Introduction. Structures are mostly designed to withstand earthquake and/or wind loads. Moment frames, reinforced concrete shear walls, and steel bracings are usually used as the lateral load resisting systems. Among them, bracing steel is an ideal solution for earthquake-resistant structural systems. It can exhibit such favorable seismic behavior characteristics as material ductility and energy absorption. However, considerable care is required in design, in order to make use of these characteristics.

Various bracing configurations have been used for different purposes, some of which are illustrated in Fig. 1. Although many new bracing configurations have been presented and studied during the last decade, the diagonal bracing is still used in steel structures of industrial and commercial buildings and transmission towers.

During the last decade, several studies have focused on the out-of-plane buckling of bracing systems, including lateral buckling, as design factors of considerable importance. In this paper, a new way of using brace elements is suggested for different bracing systems, and their properties are investigated. In addition, relationships are obtained for calculating the critical load of these bracings. Although this paper formulates and solves the problem for the general form of X-bracing systems, the idea could easily be used for other bracing systems as well in order to increase the buckling load of braces. It is because the braces in other configurations with out-of-plane buckling, from the viewpoint of stability, have no bilateral interaction and work as individual elements. Thus, the use of pretwisted bars as braces in other configurations needs no more stability analysis, and the resulting increase in the buckling load could easily be calculated referring to [1, 2]. The only remaining issue is the geometrical problem of using such elements, e.g., making proper connections to the other prismatic frame beams that are considered in the last part of the paper. Shadnam has considered production of pretwisted beams [3].

As will be shown in Sec. 1, stability of pretwisted individual elements and stability of concentric bracing systems have been hitherto considered independently. The present paper is the intersection point of the two aforementioned research trends. The paper uses the theoretical stability investigations of pretwisted beams, i.e., bending/compression beams that have experienced permanent twist (Fig. 2), to improve usual bracing systems. When the bar is pretwisted, its strong axis buckling equation is coupled with that of its weak axis. Furthermore, the buckling equation of the compression brace is coupled with that of the tension brace. Therefore, a system of four coupled differential equations is considered in this paper.

1. Literature Review. The paper actually combines two problems: one from mathematical physics (pretwisted bar stability) and the other from structural engineering (bracing stability).

The first investigation of the stability of pretwisted bars dates back to 1948 when Ziegler derived the buckling

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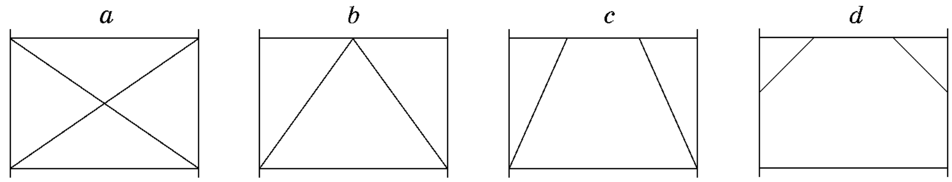

Fig. 1. Various types of bracing: X-bracing (a), V-bracing (b), eccentric bracing (c), and knee bracing (d).

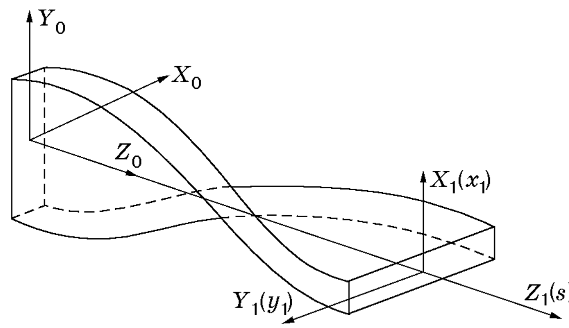

Fig. 2. Pretwisted beam.

equation for a simply supported Euler column [4]. A more detailed analysis was later performed in [5], where the governing equations were derived for two different coordinate systems and a more complete treatment of different possible boundary conditions was given. Lüscher [6] solved Ziegler's equations for the case of a cantilevered column. Frisch-Fay [7] derived Ziegler's equations and confirmed his original solution. Gupta and Rao [8] analyzed the stability of tapered and twisted Timoshenko beams, and Celep [9–11] investigated the dynamic buckling of pretwisted beams under nonconservative loads. During the last decade, Tabarrok and his colleagues derived general fourth-order coupled buckling equations of pretwisted bars [1, 2]. Recent works on the study of pretwisted rods center around the use of exact and approximate numerical techniques, such as the finite element method, to study the vibration of more complicated pretwisted beam models.

Several studies have been performed to solve the second problem. DeWolf and Pelliccione [12] and Kitipronchai and Finch [13] derived exact solutions for pin-ended diagonals connected at midspan for various T/P ratios (T and P are the tension and compression forces in the bar, respectively). Wang and Boresi [14] considered fix-ended constraints as well. Most of these studies predominantly concentrate on cross bracings that comprise identical diagonals, with one diagonal always in tension. Diagonals are either simply or rigidly attached at their ends to the framework. Stoman [15, 16] used the Rayleigh–Ritz method to obtain closed-form analytic relationships for symmetric cross-bracing systems for various values of T/P. Knapp and Dixon [17] presented closed-form solutions for the pinned and rigid cases under a variety of T/P ratios for non-midspan connected bracing systems in offshore platforms. In these studies, the external forces were assumed to act axially on the diagonal bracing. Thevendran and Wang [18] presented a numerical technique based on the energy method for obtaining simple expressions for buckling loads and effective length factors in asymmetric diagonals for pin- and fix-ended connections. On the experimental level, El-Tayem and Goel [19] provided recommendations on calculating the effective length factor for design purposes, as was drawn from full-scale specimens. Kemp and Behncke [20] performed tests on typical panels of lattice towers, to compare with test results and design codes. Segal et al. [21] considered semi-rigid connections in symmetric cross bracings and solved the resulting equations exactly.

2. Governing Equations. General behavioral relationships for mutually interactive compression and tension diagonals under symmetric conditions are derived and solved in Secs. 2 and 3. These solutions, which are the lateral displacements, are then used to obtain the lateral stiffnesses for two distinct cases, the pinned and the built-in cases, from which conditions for general loss of stiffness (overall buckling) are derived and studied.

We consider a single-story rectangular one-bay frame that is simply supported and diagonally braced to resist a horizontal force (Fig. 1a). Overall structural analysis will produce forces of equal magnitude but of opposite sign to the braces. These are marked as T and P. The applied loads may change their direction, and as a result, the diagonals change roles in resisting the loads in the sense that the tension diagonal is now in compression, whereas

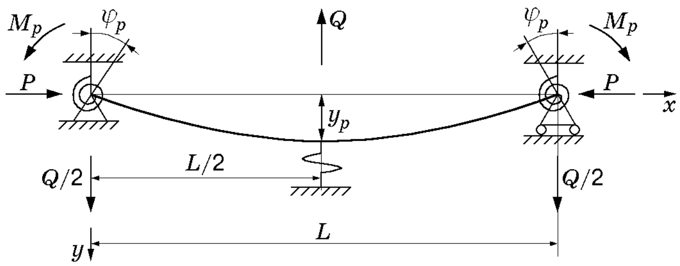

Fig. 3

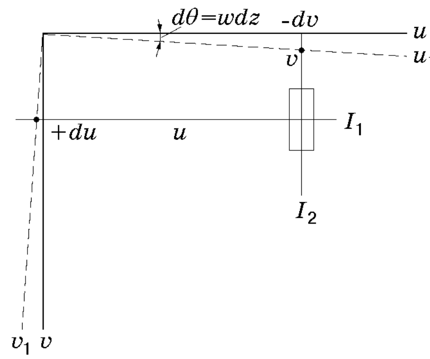

Fig. 4

Fig. 3. Compression diagonal.

Fig. 4. Cross section of a deformed bar.

the compression diagonal becomes tensile. These forces are not independent. They are internal forces that depend on the geometry and relative stiffnesses, and their derivation considers the frame as a whole rather than an isolated bracing system, therefore, consistency is maintained. At this stage, a simplified design of the compression diagonal usually implies that $T = 0$ and that the compression beam is laterally restrained by a constant spring, which is actually the lateral stiffness of the tension diagonal (Fig. 3). This paper considers the tension lateral stiffness as an integral part of a cross-bracing system for the design of the pretwisted compression diagonal.

We consider a prismatic elastic pretwisted beam of length L subjected to an axial compressive force P, lateral load Q, and bending moment M_p. The load Q is, in fact, the transverse force between the compression and tension diagonals at their point of intersection.

Let u and v be the principal axes of the system at height z, where z is measured along the bar axis from a fixed reference point coinciding with the bottom support. At $z+dz$, the principal axes change to u_1 and v_1 (Fig. 4). In the buckled state, the cross section is displaced by u and v from its initial position on the z axis.

The equilibrium conditions in bending for the nontwisted bar (nonrotating axes) are

$$EI_2 \frac{d^2u}{dz^2} + Pu = Q\frac{z}{2} + M_p, \qquad EI_1 \frac{d^2v}{dz^2} + Pv = 0, \tag{1}$$

where $0 < z < L/2$.

We now consider reference axes that rotate with the principal axes of the section. Let w be the angle of twist per unit length of the bar. Rotation of the reference axes will modify the displacements u and v, and hence, the expressions for curvature. We consider the first equation of system (1). At a distance dz, the u and v axes rotate by an angle $w\,dz$ (Fig. 4). Compared with the fixed axis system, u and v are additionally changed by

$$du = vw\,dz; \tag{2}$$

$$dv = -uw\,dz. \tag{3}$$

We obtain $d^2u/dz^2 = w\,dv/dz$ from (2) and $uw^2 + w\,dv/dz = 0$ from (3). Observing that the curvature of the line u decreases by wv (if the cross-section rotates clockwise), the first equation in (1) becomes

$$EI_2(u'' - 2v'w - w^2u) + Pu = (Qz/2 + M_p)\cos(wL/2), \tag{4}$$

where $(\cdot)' = d(\cdot)/dz$. Similarly, the second equation in (1) is written as

$$EI_1(v'' + 2u'w - w^2v) + Pv = (Qz/2 + M_p)\sin(wL/2). \tag{5}$$

It should be pointed out here that two terms are subtracted from the first equation of system (1), and two similar ones are added to the second one. By virtue of Eqs. (2) and (3), an interchange of these operations would have resulted in two independent differential equations because of the elimination of the effect of the pretwist on the curvatures of u'' and v''. In the dimensionless variables $\xi = x/L$, $U = u/x$, $wL = q$, $v^2 = PL^2/(EI_2)$, and $\beta^2 = QL^2/(EI_2)$, Eqs. (4), (5) are written as

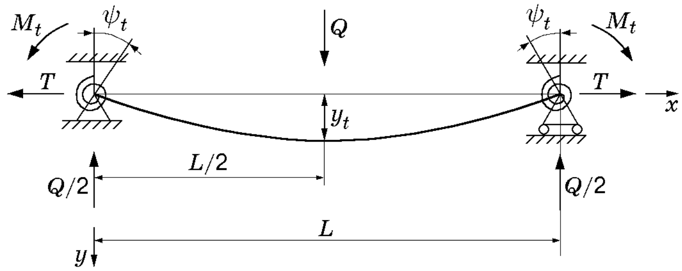

Fig. 5. Tensile diagonal.

$$\frac{d^2U}{d\xi^2} - 2q\,\frac{dV}{d\xi} + U(v^2 - q^2) = \frac{\beta^2\xi\cos(q/2)}{2}, \tag{6}$$

$$\frac{d^2V}{d\xi^2} + 2q\,\frac{dU}{d\xi} + V\Big(\frac{v^2}{2} - q^2\Big) = \frac{\beta^2\xi\sin(q/2)}{4},$$

where v is the dimensionless buckling parameter. Another representation is $\sqrt{\pi/K}$, where K is the effective length factor. The boundary conditions of Eqs. (6) are

$$U(0) = V(0) = \frac{1}{2}\,\frac{dU}{d\xi} = \frac{1}{2}\,\frac{dV}{d\xi} = 0. \tag{7}$$

It was assumed that $I_1 = 2I_2$ in deriving Eqs. (6).

3. Analytical Solution. Introducing $a = 2q$, $b = v^2 - q^2$, and $c = v^2/2 - q^2$, we obtain the auxiliary equation for system (6):

$$m^4 + (a^2 + b + c)m^2 + bc = 0. \tag{8}$$

Let m_1 and m_2 be the roots of the modified Eq. (8):

$$m^2 + fm + g = 0.$$

Then, the analytical solutions of system (6) are

$$U = A\exp(p_1\xi) + B\exp(-p_1\xi) + C\exp(p_2\xi) + D\exp(-p_2\xi) + A_0\xi + A_1,$$

$$V = Av_1\exp(p_1\xi) - Bv_1\exp(-p_1\xi) + Cv_2\exp(p_2\xi) - Dv_2\exp(-p_2\xi) + B_0\xi + B_1,$$

where $p_1 = \sqrt{m_1}$, $p_2 = \sqrt{m_2}$, $v_1 = (p_1^2 + b^2)ap_1$, and $v_2 = (p_2^2 + b^2)/(ap_2)$.

Using the method of undetermined coefficients [22], the coefficients of a particular solution could be determined as

$$A_0 = \beta^2\cos(q/2)/(2(v^2 - q^2)), \qquad B_0 = \beta^2\sin(q/2)/(2(v^2 - 2q^2)),$$

$$A_1 = 2qB_0/(v^2 - q^2), \qquad B_1 = -2qA_0/(v^2/2 - q^2).$$

Using the boundary conditions (7), the remaining coefficients A, B, C, and D, and then the value of U for $\xi = 1/2$ could be determined exactly [3].

We now consider a prismatic elastic pretwisted beam of length L subjected to a tensile force T and the same lateral load Q (Fig. 5). The differential equation for bending is similar to the differential equation for bending of the compression bar except that v^2 and β^2 should be replaced by $\alpha^2 = \sqrt{Tl^2/(EI_2)}$ and $-\beta^2$, respectively. The values of U for the tensile brace could also be obtained by going on similar steps. Defining the lateral stiffnesses k_t and k_p of the tension and compression diagonals as β^2/U_t and $-\beta^2/U_p$, respectively, where U_t and U_p are the displacements at midspan, the continuity condition $U_t = U_p$ at this point can be written as

$$k_p + k_t = 0. \tag{9}$$

4. Results. Monstrous expressions that stem from algebraic manipulations and taking derivatives were handled by an effective symbolic mathematical software (Maple 5). The results of analytical calculations can be found in [3].

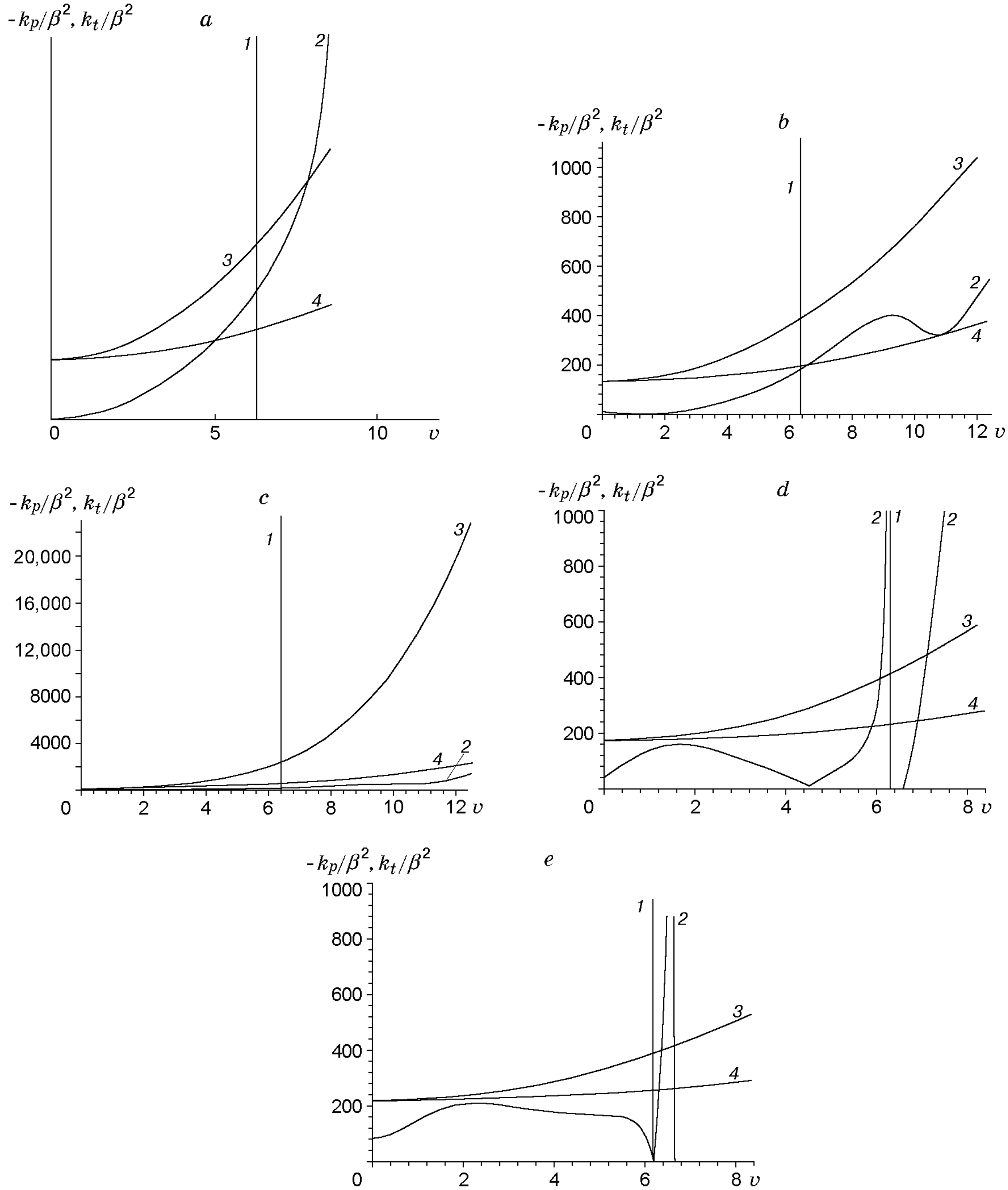

Fig. 6. Buckling of X-bracing with pinned ends for $q = 0$ [21] (a), 90° (b), 180° (c), 270° (d), and 360° (e): 1 is the boundary line, curve 2 is the dependence $-k_p/\beta^2(v)$, and curves 3 and 4 show the dependence $k_t/\beta^2(v)$ for $\alpha = v$ (3) and $\alpha = 0.5v$ (4).

The solution of Eq. (9) depends on α, v, and q only. It applies to any choice of lengths, flexural stiffnesses, and axial loads for both tension and compression diagonals.

Obviously, we have $v < 2\pi$ for $q = 0$. This is a condition that stems from the definition of Euler's buckling. For $v > 2\pi$ and $q = 0$, the antisymmetric mode is dominant [21]. Figures 6 and 7 show the graphical solution of Eq. (9) for different values of q and various ratios α/v. Curve 2 is a plot of $-k_p/\beta^2(v)$. Curves 3 and 4 represent $k_t/\beta^2(v)$ for various ratios α/v. The solutions of Eq. (9) are the points of intersection of curve 2 with curves 3 and 4. Line 1 is the boundary line to the left of which the solutions have a buckling mode that is a symmetric

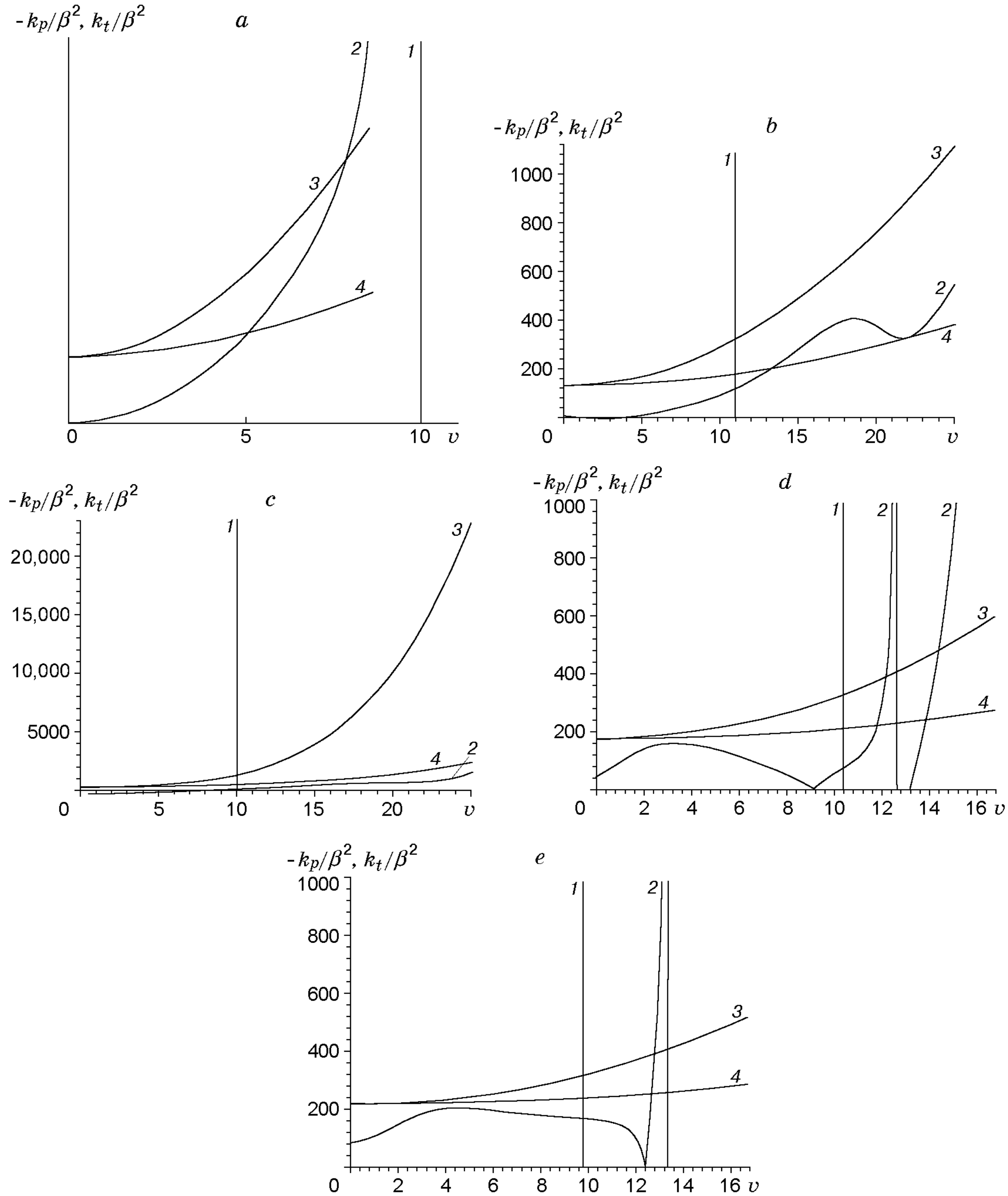

Fig. 7. Buckling of X-bracing with clamped ends (notation the same as in Fig. 6).

first mode. The solutions to the right of line 1 have unstable symmetric modes. The position of line 1 was taken from [2].

In the pinned-pinned cross bracing, if $\alpha = v$ (usual practice), the solution is on the right of line 1 for $q = 0$. The first buckling mode is, therefore, antisymmetric [21]. In the same case, the solution is on the right of line 1 for $q = 90$ and 180° and on the left of it for $q = 270$ and 360°. This means that the first buckling mode is symmetric only for $q = 270$ and 360°.

For $\alpha = 0.5v$, the solution of the pinned-pinned case is on the left of line 1 for $q = 0$, 180, and 360° and on the right of it for $q = 90$ and 270°. This means that the first buckling mode is antisymmetric only for $q = 90$ and 270°. It is obvious that antisymmetric modes prevail when the ratio $T/P = \alpha/v$ increases. Furthermore, for

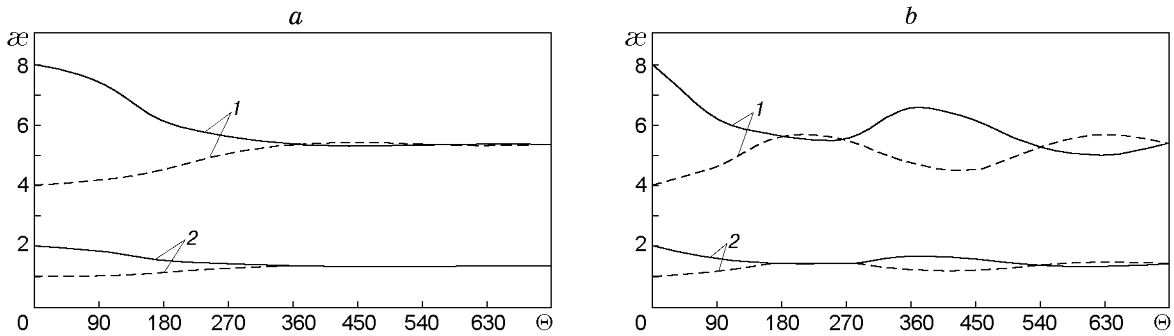

Fig. 8. Strength ratio versus the total twist for a pretwisted beam for cross bracing with pinned (a) and clamped (b) ends: curves 1 and 2 refer to asymmetric and symmetric buckling; the solid and dashed curves refers to the first and second modes, respectively.

$q = 180°$, the antisymmetric mode is dominant even for low values of α/v, because the cross section has rotated by $q = 90°$ at midspan.

By solving the associate homogenous differential equations, the buckling loads for different total twist of the beam can be calculated. The results are actually the same as those presented by Tabarrok et al. [2] for the ratio of the moments of inertia $\rho = I_2/I_1 = 2$. The results of these calculations are plotted in Fig. 8. The abscissa axis is the total twist Θ, and the ordinate axis is æ (the ratio of the buckling load of a pretwisted beam to the buckling load of a nontwisted beam). The lower pair of curves refers to symmetric buckling, and the upper pair to asymmetric buckling. This plot exhibits the expected increase in the buckling strength for the first mode as the angle of twist is increased and a more rapid decrease in the buckling strength of the second mode.

Geometric Considerations. In X-bracing, the braces should be connected properly in the middle in order to transmit forces. But when the braces are pretwisted, the problem is complicated, and one should take into account the angle of twist of braces at the intersection position in order to design the intersection.

As can be seen from Fig. 8a, for the pinned-pinned case and close to 360° total twist, there is a confluence of the first and the second buckling modes. This means from the practical point of view that no more strength can be achieved by pretwist. In the case of symmetric X-bracings, in which the intersection is in the middle, the section at the intersection will twist by 180° if one end of the beam is twisted by 360°. The same is true for the other beam of X-bracing. This means that, at the intersection, one has to connect 180° twisted cross sections to each other. As the constructional sections are composed of perpendicular parts, the beams could safely be connected to each other.

On the other extreme, in the clamped-clamped case, as can be derived from Fig. 8b, the best pretwist occurs somewhere between 180 and 270°. From the constructional point of view, as the usual sections are composed of perpendicular parts, the beams could not be connected properly to each other. No proper connection for a symmetric case could be designed unless the overall twist angle is taken a multiple of 180°. Thus, the only proper total twist for a symmetric case can be 180°. In this case, the braces are twisted by 90° in the middle section, and the braces can be connected to each other properly.

For intermediate cases, i.e., semirigid connections, similar diagrams of the strength ratio versus the total twist should be constructed, and then practical recommendations can be given.

It should be noted that the proper twist angle depends significantly on the geometry for the asymmetric topology of the construction.

6. Conclusions. Equation (9) is a generic expression for evaluating critical loads for arbitrary lengths of diagonals, flexural stiffnesses, and tensile and compression loads.

For all cases analyzed, the effect of natural twist is to increase the first buckling load and diminish the second one. The plots of the strength ratios (see Fig. 8) versus the total angle of twist indicate a coalescence of the first and second buckling loads as the total angle of twist increases. Depending upon the type ot boundary conditions and principal moments of inertia of the cross sections, the plots show that the strength of the X-bracings could be increased by 40 to more than 100%. However, the present paper does not deal with the problems related to the production of such sections and is just the first step.

The authors hope this idea some day could efficiently serve design engineers in real design conditions.

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